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New variant of $N = 4$ superconformal mechanics

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Abstract: Proceeding from a nonlinear realization of the most general N=4, d=1 superconformal symmetry, associated with the supergroup $D(2,1;\alpha)$, we construct a new model of nonrelativistic N=4 superconformal mechanics. In the bosonic sector it combines the worldline dilaton with the fields parametrizing the R-symmetry coset $S^2 \sim \mathrm{SU}(2)/\mathrm{U}(1)$. We present invariant off-shell N=4 and N=2 superfield actions for this system and show the existence of an independent N=4 superconformal invariant which extends the dilaton potential. The extended supersymmetry requires this potential to be accompanied by a d=1 WZW term on S^2. We study the classical dynamics of the bosonic action and the geometry of its sigma-model part. It turns out that the relevant target space is a cone over S^2 for any non-zero $\alpha \neq \pm 1/2$. The constructed model is expected to be related to the 'relativistic' N=4 mechanics of the $\mathrm{AdS}_2 \times S^2$ superparticle via a nonlinear transformation of the fields and the time variable.

Keywords: Field Theories in Lower Dimensions, Conformal and W Symmetry, Extended Supersymmetry.

Contents

1. Introduction

Models of conformal and superconformal quantum mechanics [1]– [14] are important basically due to their role as 'conformal field theories' in the $\mathrm{AdS}_2/\mathrm{CFT}_1$ version of the AdS/CFT correspondence [15], the fact that they describe the near-horizon dynamics of the black-hole solutions of supergravity, and their tight relation with integrable Calogero-Moser type systems.

The recent renewal of interest in (super)conformal mechanics models was mainly triggered by the observation [6] that the radial motion of a superparticle at zero angular momentum in the near-horizon region of the extreme Reissner-Nordström black hole is governed by a modified ('new' or 'relativistic') version of conformal mechanics. The corresponding background is $\mathrm{AdS}_2 \times S^2$, with the radial coordinate of the superparticle parametrizing its AdS_2 part. Since the full isometry of this background is the supergroup $\mathrm{SU}(1,1|2)$, it was suggested in [6] that the full (radial and angular) dynamics of this superparticle is described by a 'new' variant of $N{=}4$ superconformal mechanics[1] containing three physical bosonic fields in its supermultiplet (the radial AdS_2 coordinate and two angular coordinates parametrizing S^2). Yet, even the 'old' conformal mechanics [1] has not been extended in full generality to an $N{=}4$ superconformal mechanics (with additional angular fields). The $N{=}4$ superconformal mechanics constructed in [5] contains only one bosonic physical field in its off-shell supermultiplet (namely the dilaton associated with the generator of spontaneously broken dilatations). In this sense the model represents a minimal $N{=}4$ extension of the standard one-component conformal mechanics of ref. [1].

[1]By N we always count *real* $d{=}1$ Poincaré supercharges. This convention differs from the nomenclature due to ref. [16] in which N denotes the number of *complex* $d{=}1$ Poincaré supercharges.

An attempt to define the most general superconformal mechanics on SU$(1,1|2)$ using the nonlinear realization approach was undertaken in [7]. However, neither an invariant action nor the solution of the irreducibility constraints on the Goldstone superfields were presented.

A Green-Schwarz-type action for the $\mathrm{AdS}_2 \times S^2$ superparticle was constructed in [17]. After properly gauge-fixing kappa-symmetry and worldline diffeomorphisms the corresponding action can be treated as an $N{=}4$ superconformal mechanics action extending that of the 'new' conformal mechanics. However, the $N{=}4$ superconformal symmetry in such an action is non-manifest and closes only on shell, which is typical for the gauge-fixed Green-Schwarz-type actions.

There are two basic ways of constructing superconformally invariant actions in one dimension. One of them proceeds from the most general $N{=}1$ supersymmetric sigma model on the $d{=}1$ worldline of the particle (in general supplemented by a coupling to an external electromagnetic potential). Then one looks for the restrictions the extra superconformal invariance imposes on the geometry of such a sigma model (see e.g. [12]–[14]). In this approach one as a rule deals with component fields (or at most $N{=}1$ superfields), and the extra Poincaré and conformal supersymmetries are not manifest. The other approach, pioneered in [5] and later employed in [7], starts from nonlinear realizations of appropriate superconformal symmetries in terms of a set of worldline Goldstone superfields. It is a natural generalization of the approach of ref. [4] where conformal mechanics [1] was shown to have a natural description in terms of a nonlinear realization of SO$(1,2)$, the conformal group in one dimension. In this way the aforementioned minimal $N{=}4$ superconformal mechanics, associated with the supergroup SU$(1,1|2)$ and having only a dilaton field in the bosonic sector, was found and studied [5]. Besides keeping manifest and off-shell the complete set of $d{=}1$ Poincaré supersymmetries, this second approach has the advantage that it automatically generates the correct irreducibility conditions for the basic Goldstone superfields and ensures a geometric interpretation for their components (the bosonic and fermionic $d{=}1$ Goldstone fields) as parameters of the relevant supercosets.

In the present paper, following the nonlinear realization approach of [5], we construct the most general $N{=}4$ superfield extension of the 'old' conformal mechanics with additional angular fields in the bosonic sector. We start with the exceptional supergroup $D(2,1;\alpha)$ [18] as the most general $N{=}4$ superconformal group in one dimension. After the appropriate choice of the supercoset, the only Goldstone $N{=}4$ superfields surviving the inverse Higgs procedure [20] are the superdilaton and the two parameters of the coset SU(2)/U(1) (a second SU(2) $\subset D(2,1;\alpha)$ is placed into the stability subgroup and acts on the fermions only). For these three superfields there naturally appear simple irreducibility conditions. Surprisingly, the latter prove to be identical to those used in [21] for constructing the off-shell $N{=}4$ supersymmetric $1d$ sigma model action with a bosonic target manifold of dimension $3k$. It is easy to solve these constraints through an unconstrained prepotential and to construct two independent superfield invariants which in components give rise to the kinetic term and a scalar potential term. The latter is necessarily accompanied by a worldline WZW term on S^2 which is nothing but the coupling of the charged particle to a Dirac magnetic monopole (see e.g. [22]). Besides the $N = 4$ superfield form of the invariant action we present its $N = 2$ superfield form. As was recently shown [23], the 'old' and 'new' conformal mechanics models are actually related by a nonlinear transformation of the time variable and the coordinate fields (it is a particular case of the 'holographic' transformation introduced in [24]). We expect that this equivalence extends to the case we are dealing with. Namely, at least for the special case of $D(2,1;\alpha{=}-1) \sim \mathrm{SU}(1,1|2) \propto \mathrm{SU}(2)$ we surmise the existence of a nonlinear transformation which maps the $N{=}4$ superconformal mechanics we shall presently construct to the 'new' $N{=}4$ superconformal mechanics of the $\mathrm{AdS}_2 \times S^2$ superparticle. On shell its action should coincide with the gauge-fixed action of ref. [17].[2]

2. $D(2,1;\alpha)$ supergroup as $N = 4$ superconformal group in $d = 1$ and its nonlinear realization

We use the standard definition of the superalgebra $D(2,1;\alpha)$ [18] with the notations of ref. [26]. It contains the following generators.

Spinor generators:

$$Q^{Ai'i}\,, \qquad \overline{(Q^{Ai'i})} = \epsilon_{ik}\epsilon_{i'k'}Q^{Ak'k}\,, \qquad (A,i,i'=1,2)\,. \tag{2.1}$$

Bosonic generators:

$$T_2^{AB}\,, \qquad T^{i'j'}\,, \qquad T_1^{ij}\,. \tag{2.2}$$

The indices A, i and i' refer to fundamental representations of the mutually commuting $\mathrm{sl}(2,\mathbb{R})$ and two $\mathrm{su}(2)$ algebras which form the bosonic sector of $D(2,1;\alpha)$. The bosonic generators obey the following conjugation rules:

$$\left(T_2^{AB}\right)^\dagger = T_2^{AB}\,, \qquad \left(T^{11}\right)^\dagger = T^{22}\,,\ \left(T^{12}\right)^\dagger = -T^{12}\,, \qquad \left(T_1^{11}\right)^\dagger = T_1^{22}\,,\ \left(T_1^{12}\right)^\dagger = -T_1^{12}\,, \tag{2.3}$$

and satisfy the commutation relations (common for all T):

$$\left[T^{ab}, T^{cd}\right] = -i\left(\epsilon^{ac}T^{bd} + \epsilon^{bd}T^{ac}\right), \tag{2.4}$$

where a, b stand for any sort of doublet indices (A, i or i'). The commutator of any generator T^a with Q^b (other indices of $Q^{Aii'}$ being suppressed) reads:

$$\left[T^{ab}, Q^c\right] = -\frac{i}{2}\left(\epsilon^{ac}Q^b + \epsilon^{bc}Q^a\right). \tag{2.5}$$

At last, the anticommutator of two fermionic generators is given by [3]

$$\left\{Q^{Ai'i}, Q^{Bk'k}\right\} = -2\left(\alpha\epsilon^{AB}\epsilon^{i'k'}T_1^{ik} + \epsilon^{ik}\epsilon^{i'k'}T_2^{AB} - (1+\alpha)\epsilon^{AB}\epsilon^{ik}T^{i'k'}\right). \tag{2.6}$$

Here, α is an arbitrary real parameter. At $\alpha = 0$ and $\alpha = -1$ one of the $\mathrm{su}(2)$ algebras decouples and we recover the superalgebra $\mathrm{su}(1,1|2) \oplus \mathrm{su}(2)$. The superalgebra $D(2,1;1)$

[2] At the on-shell hamiltonian level, the issue of canonical equivalence between some versions of these two, at first sight different, N=4 superconformal mechanics systems is studied in [25].

[3] We use the following convention for the skew-symmetric tensor ϵ: $\epsilon_{ij}\epsilon^{jk} = \delta_i^k\,, \qquad \epsilon_{12} = \epsilon^{21} = 1\,.$

is isomorphic to $osp(4^*|2)$. There are some different choices of the parameter α which lead to isomorphic algebras $D(2,1;\alpha)$ [18].

For what follows it is convenient to pass to another notation,

$$
\begin{aligned}
& P \equiv T_2^{22}\,, \qquad K \equiv T_2^{11}\,, \qquad D \equiv -T_2^{12}\,, \\
& T \equiv T^{22}\,, \qquad \overline{T} \equiv T^{11}\,, \qquad T_3 \equiv T^{12}\,, \qquad V \equiv T_1^{22}\,, \qquad \overline{V} \equiv T_1^{11}\,, \qquad V_3 \equiv T_1^{12}\,, \\
& Q^i \equiv -Q^{21i}\,, \qquad \overline{Q}^i \equiv -Q^{22i}\,, \qquad S^i \equiv Q^{11i}\,, \qquad \overline{S}^i \equiv Q^{12i}\,.
\end{aligned} \tag{2.7}
$$

One may check that P and $Q^i, \overline{Q}^i$ constitute the $N=4, d=1$ Poincaré superalgabra. The generators D, K and $S^i, \overline{S}^i$ stand for $d=1$ dilatations, special conformal transformations and conformal supersymmetry, respectively.

We shall construct a nonlinear realization of the superconformal group $D(2,1;\alpha)$ on the coset superspace parametrized as

$$
g = e^{itP} e^{\theta_i Q^i + \bar\theta^i \overline{Q}_i} e^{\psi_i S^i + \bar\psi^i \overline{S}_i} e^{izK} e^{iuD} e^{i\phi V + i\bar\phi \overline{V}}\,. \tag{2.8}
$$

The coordinates $t, \theta_i, \bar\theta^i$ parametrize the $N=4, d=1$ superspace. All other supercoset parameters are Goldstone $N=4$ superfields. The stability subgroup contains a U(1) subgroup of the group SU(2) realized on the doublet indices i, hence the Goldstone superfields $\phi, \bar\phi$ parametrize the coset SU(2)/ U(1). Another SU(2) is entirely placed in the stability subgroup and acts only on fermionic Goldstone superfields and θ's, mixing them with their conjugates. With our choice of the SU(2) coset we are led to assume that $\alpha \neq 0$. We could equivalently choose another SU(2) to be nonlinearly realized and the first one to belong to the stability subgroup, then the restriction $\alpha \neq -1$ would be imposed instead.

The left-covariant Cartan one-form Ω with values in the superalgebra $D(2,1;\alpha)$ is defined by the standard relation

$$
g^{-1}\, d\, g = \Omega\,. \tag{2.9}
$$

In what follows we shall need to know the explicit structure of several important one-forms in the expansion of Ω over the generators (2.7),

$$
\begin{aligned}
\omega_P &= ie^{-u} d\tilde t \equiv ie^{-u}\left[dt + i\left(\theta_i d\bar\theta^i + \bar\theta^i d\theta_i\right)\right], \\
\omega_1^Q &= \frac{1}{\sqrt{1+\Lambda\overline{\Lambda}}}\left[\hat\omega_1^Q + \overline{\Lambda}\hat\omega_2^Q\right], \quad \omega_2^Q = \frac{1}{\sqrt{1+\Lambda\overline{\Lambda}}}\left[\hat\omega_2^Q - \Lambda\hat\omega_1^Q\right], \\
\bar\omega_Q^1 &= \frac{1}{\sqrt{1+\Lambda\overline{\Lambda}}}\left[\hat{\bar\omega}_Q^1 + \Lambda\hat{\bar\omega}_Q^2\right], \quad \bar\omega_Q^2 = \frac{1}{\sqrt{1+\Lambda\overline{\Lambda}}}\left[\hat{\bar\omega}_Q^2 - \overline{\Lambda}\hat{\bar\omega}_Q^1\right], \\
\omega_D &= idu - 2\left(\bar\psi^i d\theta_i + \psi_i d\bar\theta^i\right) - 2iz d\tilde t\,, \\
\omega_V &= \frac{1}{1+\Lambda\overline{\Lambda}}\left[id\Lambda + \hat\omega_V + \Lambda^2\hat{\bar\omega}_V - \Lambda\hat\omega_{V_3}\right], \quad \bar\omega_V = \frac{1}{1+\Lambda\overline{\Lambda}}\left[id\overline{\Lambda} + \hat{\bar\omega}_V + \overline{\Lambda}^2\hat\omega_V + \overline{\Lambda}\hat\omega_{V_3}\right], \\
\omega_{V_3} &= \frac{1}{1+\Lambda\overline{\Lambda}}\left[i\left(\Lambda d\overline{\Lambda} - \overline{\Lambda} d\Lambda\right) + \left(1-\Lambda\overline{\Lambda}\right)\hat\omega_{V_3} - 2\left(\Lambda\hat{\bar\omega}_V - \overline{\Lambda}\hat\omega_V\right)\right].
\end{aligned} \tag{2.10}
$$

Here

$$\hat\omega_i^Q = e^{-\frac{1}{2}u}\left(d\theta_i - \psi_i d\tilde t\right), \qquad \hat{\bar\omega}_Q^i = e^{-\frac{1}{2}u}\left(d\bar\theta^i - \bar\psi^i d\tilde t\right),$$
$$\hat\omega_V = 2\alpha\left[\psi_2 d\bar\theta^1 - \bar\psi^1\left(d\theta_2 - \psi_2 d\tilde t\right)\right], \qquad \hat{\bar\omega}_V = 2\alpha\left[\bar\psi^2 d\theta_1 - \psi_1\left(d\bar\theta^2 - \bar\psi^2 d\tilde t\right)\right],$$
$$\hat\omega_{V_3} = 2\alpha\left[\psi_1 d\bar\theta^1 - \bar\psi^1 d\theta_1 - \psi_2 d\bar\theta^2 + \bar\psi^2 d\theta_2 + \left(\bar\psi^1\psi_1 - \bar\psi^2\psi_2\right)d\tilde t\right], \tag{2.11}$$

and

$$\Lambda = \frac{\tan\sqrt{\phi\bar\phi}}{\sqrt{\phi\bar\phi}}\phi, \qquad \overline\Lambda = \frac{\tan\sqrt{\phi\bar\phi}}{\sqrt{\phi\bar\phi}}\bar\phi. \tag{2.12}$$

The semi-covariant (fully covariant only under Poincaré supersymmetry) spinor derivatives are defined by

$$D^i = \frac{\partial}{\partial\theta_i} + i\bar\theta^i\partial_t, \qquad \overline D_i = \frac{\partial}{\partial\bar\theta^i} + i\theta_i\partial_t, \qquad \left\{D^i, \overline D_j\right\} = 2i\delta^i_j\partial_t. \tag{2.13}$$

As the next step, we impose the inverse Higgs constraints [20]

$$\omega_D = 0, \qquad \omega_V| = \bar\omega_V| = 0, \tag{2.14}$$

where $|$ means spinor projections. These constraints are manifestly covariant under the whole supergroup. They allow one to eliminate the Goldstone spinor superfields and the superfield z as the spinor and t-derivatives, respectively, of the residual bosonic Goldstone superfields $u, \Lambda, \bar\Lambda$ and simultaneously imply some irreducibility constraints for the latter:

$$\begin{aligned} &iD^i u = -2\bar\psi^i, && i\overline D_i u = -2\psi_i, && \dot u = 2z, \\ &iD^1\Lambda = 2\alpha\Lambda\left(\bar\psi^1 + \Lambda\bar\psi^2\right), && iD^2\Lambda = -2\alpha\left(\bar\psi^1 + \Lambda\bar\psi^2\right), && \\ &i\overline D_1\Lambda = 2\alpha\left(\psi_2 - \Lambda\psi_1\right), && i\overline D_2\Lambda = 2\alpha\Lambda\left(\psi_2 - \Lambda\psi_1\right). && \end{aligned} \tag{2.15}$$

To understand the meaning of these constraints, let us pass to the new variables

$$q = e^{\alpha u}\frac{1-\Lambda\overline\Lambda}{1+\Lambda\overline\Lambda}, \qquad \lambda = e^{\alpha u}\frac{\Lambda}{1+\Lambda\overline\Lambda}, \qquad \bar\lambda = e^{\alpha u}\frac{\overline\Lambda}{1+\Lambda\overline\Lambda}. \tag{2.16}$$

In these variables the constraints are rewritten as

$$D^1\lambda = \overline D^1\lambda = 0, \qquad D^2\bar\lambda = \overline D^2\bar\lambda = 0,$$
$$D^1 q = D^2\lambda, \qquad D^2 q = -D^1\bar\lambda,\ \overline D_1 q = \overline D_2\bar\lambda, \qquad \overline D_2 q = -\overline D_1\lambda. \tag{2.17}$$

After introducing a new $N=4$ vector superfield V^{ij} subject to $V^{ij} = V^{ji}$ and $\overline{V^{ik}} = \epsilon_{ii'}\epsilon_{kk'}V^{i'k'}$ via

$$V^{11} = -i\sqrt2\,\lambda, \qquad V^{22} = i\sqrt2\,\bar\lambda, \qquad V^{12} = \frac{i}{\sqrt2}q,$$
$$V^2 \equiv V^{ik}V_{ik} = q^2 + 4\lambda\bar\lambda, \tag{2.18}$$

the constraints (2.17) can be brought in the manifestly SU(2)-symmetric suggestive form

$$D^{(i}V^{jk)} = 0, \qquad \overline D^{(i}V^{jk)} = 0. \tag{2.19}$$

The superfield V^{ik} subject to (2.19) is recognized as the one employed in [21] for constructing a general off-shell $N=4$ supersymmetric $1d$ sigma model with bosonic target manifold of dimension $3k$.[4] The constraints (2.19) leave in V^{ik} the following independent superfield projections:

$$V^{ik}\,, \qquad D^i V^{kl} = -\frac{1}{3}(\epsilon^{ik}\chi^l + \epsilon^{il}\chi^k)\,, \qquad \bar{D}^i V^{kl} = \frac{1}{3}(\epsilon^{ik}\bar{\chi}^l + \epsilon^{il}\bar{\chi}^k)\,, \qquad D^i \bar{D}^k V_{ik}\,, \tag{2.20}$$

where

$$\chi^k \equiv D^i V^k_i\,, \qquad \bar{\chi}_k = \overline{\chi^k} = \bar{D}_i V^i_k\,. \tag{2.21}$$

Due to (2.19), all other superfield projections are vanishing or are expressed as time derivatives of (2.20), e.g.[5]

$$(D)^2 V^{ik} = (\bar{D})^2 V^{ik} = [D,\bar{D}] V^{ik} = 0\,, \qquad D^{(i}\bar{\chi}^{k)} = \bar{D}^{(i}\chi^{k)} = 3i\dot{V}^{ik}\,. \tag{2.22}$$

The spinor superfields χ^i, $\bar{\chi}^k$ are related to the original Goldstone fermionic superfields $\psi^i, \bar{\psi}^k$ by

$$\chi^i = 2i\alpha V^{ik}\bar{\psi}_k\,, \qquad \bar{\chi}^i = -2i\alpha V^{ik}\psi_k\,. \tag{2.23}$$

Note that the treatment of V^{ik} in the $N=4$ superconformal mechanics context is somewhat different from the one adopted in [21, 27]. In our case the superfield V^{ik} provides an example of the construction of a linear representation of SU(2) symmetry in terms of its nonlinear realization. As is seen from the identification (2.16), (2.18), it is entirely of Goldstone nature: its angular part $V^{ik}/|V|$ is related by an equivalence transformation to the SU(2)/U(1) Goldstone superfields $\Lambda, \bar{\Lambda}$ while the norm $|V| = \sqrt{q^2 + 4\lambda\bar{\lambda}} = 1 + 2\alpha\, u + \cdots$ is related to the dilaton u and is non-vanishing for vanishing fields.

Important corollaries of eqs. (2.19) are the relations

$$(D)^2 \left(V^2\right)^{-1/2} = (\overline{D})^2 \left(V^2\right)^{-1/2} = [D,\overline{D}] \left(V^2\right)^{-1/2} = 0 \tag{2.24}$$

and

$$D_i \,\frac{V^{ij}}{\left(V^2\right)^{3/2}} = \overline{D}_i \,\frac{V^{ij}}{\left(V^2\right)^{3/2}} = 0\,. \tag{2.25}$$

The constraints (2.19) can be solved in terms of an unconstrained prepotential $\mathcal{W}$,

$$V^{ij} = D^{(i}\overline{D}^{j)}\mathcal{W}\,, \qquad \bar{\mathcal{W}} = -\mathcal{W}\,, \tag{2.26}$$

which is defined up to a gauge freedom

$$\mathcal{W}\,' = \mathcal{W} + (D)^2\omega - (\bar{D})^2\bar{\omega} + [D,\bar{D}]\sigma\,, \tag{2.27}$$

where ω, $\bar{\omega}$ and $\sigma = -\bar{\sigma}$ are superfield gauge parameters.

[4]The same off-shell $N=4, d=1$ supermultiplet was independently considered in [27] and later rediscovered in [14].

[5]We use the notation: $(D)^2 = D^i D_i$, $(\bar{D})^2 = \bar{D}_i\bar{D}^i$, $[D,\bar{D}] = [D^i, \bar{D}_i]$.

Let us quote the transformation properties of the $N=4$ superspace coordinates and the basic Goldstone superfields under the transformations of conformal supersymmetry. They are generated by acting on the coset element (2.8) from the left by the element

$$g_0 = e^{\epsilon_i S^i + \bar{\epsilon}^i \bar{S}_i} \in D(2,1;\alpha)\,. \tag{2.28}$$

Hereafter, when summing over doublet indices we assume them to stay in a natural position; the Grassmann coordinates and their conjugates carry lower case and upper case indices, respectively. We use as short-hand notation $\bar{\psi}\cdot\xi = \bar{\psi}^i\xi_i = -\xi\cdot\bar{\psi}$. The contraction of spinors of equal kind is defined as $a\cdot b = a^i b_i$, $\bar{a}\cdot\bar{b} = \bar{a}_i\bar{b}^i$.

The sought transformations are

$$\begin{aligned}
\delta t &= -it\left(\epsilon\cdot\bar{\theta} + \bar{\epsilon}\cdot\theta\right) + (1+2\alpha)\theta\cdot\bar{\theta}\left(\epsilon\cdot\bar{\theta} - \bar{\epsilon}\cdot\theta\right),\\
\delta\theta_i &= \epsilon_i t - 2i\alpha\theta_i(\theta\cdot\bar{\epsilon}) + 2i(1+\alpha)\theta_i(\bar{\theta}\cdot\epsilon) - i(1+2\alpha)\epsilon_i(\theta\cdot\bar{\theta})\,,\\
\delta u &= -2i\left(\epsilon\cdot\bar{\theta} + \bar{\epsilon}\cdot\theta\right),\\
\delta\Lambda &= 2i\alpha\left[\theta_2\bar{\epsilon}^1 - \bar{\theta}^1\epsilon_2 + \left(\bar{\theta}^2\epsilon_1 - \theta_1\bar{\epsilon}^2\right)\Lambda^2 + \left(\bar{\theta}^1\epsilon_1 - \theta_1\bar{\epsilon}^1 + \theta_2\bar{\epsilon}^2 - \bar{\theta}^2\epsilon_2\right)\Lambda\right].
\end{aligned} \tag{2.29}$$

Under these, the $N=4$ superspace integration measure $dtd^4\theta$ is transformed as

$$\delta\, dtd^4\theta = -2i(\epsilon\cdot\bar{\theta} + \bar{\epsilon}\cdot\theta)\, dtd^4\theta\,. \tag{2.30}$$

The transformations of V^{ij} defined in (2.18) can be written in the following manifestly covariant fashion,

$$\delta V^{ij} = -2i\alpha\left[(\epsilon\cdot\bar{\theta} + \bar{\epsilon}\cdot\theta)V^{ij} + (\epsilon^{(i}\bar{\theta}_k - \bar{\epsilon}_k\theta^{(i})V^{j)k} + (\epsilon_k\bar{\theta}^{(i} - \bar{\epsilon}^{(i}\theta_k)V^{j)k}\right]. \tag{2.31}$$

As a consequence,

$$\delta V^2 = -4i\alpha\,(\epsilon\cdot\bar{\theta} + \bar{\epsilon}\cdot\theta)\,V^2\,. \tag{2.32}$$

The covariant derivatives D^i, $\bar{D}_i$ transform as

$$\delta D^i = i\left[(2+\alpha)(\epsilon\cdot\bar{\theta}) + \alpha(\theta\cdot\bar{\epsilon})\right]D^i - 2i(1+\alpha)(\bar{\epsilon}\cdot\bar{\theta})\bar{D}^i - 2i\alpha\left[\epsilon^{(i}\bar{\theta}_{k)} + \theta^{(i}\bar{\epsilon}_{k)}\right]D^k\,, \tag{2.33}$$

$$\delta\bar{D}_i = i\left[(2+\alpha)(\bar{\epsilon}\cdot\theta) + \alpha(\bar{\theta}\cdot\epsilon)\right]\bar{D}_i - 2i(1+\alpha)(\theta\cdot\epsilon)D_i - 2i\alpha\left[\epsilon_{(i}\bar{\theta}_{k)} + \theta_{(i}\bar{\epsilon}_{k)}\right]\bar{D}^k\,. \tag{2.34}$$

From these transformations it follows, in particular, that chiral $N=4, d=1$ superfields exist only at $\alpha=-1$, i.e. in the case of the supergroup $\mathrm{SU}(1,1|2)$.

The prepotential $\mathcal{W}$ can be shown to have the following simple transformation rule,

$$\delta\mathcal{W} = -2i(1+\alpha)\,(\epsilon\cdot\bar{\theta} + \bar{\epsilon}\cdot\theta)\,\mathcal{W}\,. \tag{2.35}$$

It is straightforward to check that the transformation laws (2.33), (2.34) and (2.35) for any $\alpha \neq 0$ imply for V^{ik} in (2.26) precisely the transformation (2.31).

For completeness, we also give the variations of the $N=4, d=1$ superspace coordinates under the $N=4, d=1$ Poincaré supergroup,

$$\delta t = i\left(\theta\cdot\bar{\varepsilon} - \varepsilon\cdot\bar{\theta}\right), \qquad \delta\theta_i = \varepsilon_i\,, \qquad \delta\bar{\theta}^i = \bar{\varepsilon}^i\,. \tag{2.36}$$

All our superfields are scalars under the latter transformations. Since all other $D(2,1;\alpha)$ transformations appear in the anticommutator of the conformal and the Poincaré supersymmetry generators, it is sufficient to require invariance under these two supersymmetries, when constructing invariant actions for the considered system.

3. Invariant actions in $N=4$ and $N=2$ superspaces

We shall construct invariant actions for our variant of $N=4$ superconformal mechanics both in $N=4$ and $N=2$ superspaces.

Due to the basic constraints (2.17), derivatives of each $N=4$ superfield with respect to, say, $\theta_1 \equiv \xi, \bar\theta^1 = \bar\xi$ can be expressed as derivatives with respect to $\theta_2 \equiv \theta, \bar\theta^2 = \bar\theta$ of other superfields. Therefore, only the $\xi = \bar\xi = 0$ components of each $N=4$ superfield are independent $N=2$ superfields. Let us denote these independent superfields as

$$q| = v\,, \qquad \lambda| = \rho\,, \qquad \bar\lambda| = \bar\rho\,, \qquad \mathcal{D}\bar\rho = \overline{\mathcal{D}}\rho = 0\,, \tag{3.1}$$

where $|$ means restriction to $\xi = \bar\xi = 0$, and $\mathcal{D}$, $\overline{\mathcal{D}}$ are spinor derivatives with respect to $\theta \equiv \theta_2$ and $\bar\theta \equiv \bar\theta^2$:

$$\mathcal{D} = \frac{\partial}{\partial\theta} + i\bar\theta\partial_t\,, \qquad \overline{\mathcal{D}} = \frac{\partial}{\partial\bar\theta} + i\theta\partial_t\,, \qquad \left\{\mathcal{D},\overline{\mathcal{D}}\right\} = 2i\partial_t\,. \tag{3.2}$$

The transformations of the implicit $N=2$ Poincaré supersymmetry completing the explicit one to the full $N=4$ have the following form in terms of these $N=2$ superfields:

$$\delta v = -\varepsilon\overline{\mathcal{D}}\bar\rho - \bar\varepsilon\mathcal{D}\rho\,, \qquad \delta\rho = \varepsilon\overline{\mathcal{D}}v\,, \qquad \delta\bar\rho = \bar\varepsilon\mathcal{D}v\,. \tag{3.3}$$

The transformations of the two R-symmetry SU(2) groups are realized as

$$\begin{array}{ll} T,\overline{T} \text{ transformations} & V,\overline{V} \text{ transformations} \\ \delta v = -a\theta\overline{\mathcal{D}}\bar\rho - \bar a\bar\theta\mathcal{D}\rho\,, & \delta v = -2(b\bar\rho + \bar b\rho) + b\bar\theta\overline{\mathcal{D}}\bar\rho + \bar b\theta\mathcal{D}\rho\,, \\ \delta\rho = a\theta\overline{\mathcal{D}}v\,, & \delta\rho = bv - b\bar\theta\overline{\mathcal{D}}v\,, \\ \delta\bar\rho = \bar a\bar\theta\mathcal{D}v\,, & \delta\bar\rho = \bar bv - \bar b\theta\mathcal{D}v\,, \end{array} \tag{3.4}$$

while the conformal supersymmetry $S^2, \overline{S}_2$ transformations act as

$$\begin{aligned} \delta v &= 2i\alpha\left(\theta\bar\epsilon + \bar\theta\epsilon\right) - \epsilon\left(t + i\theta\bar\theta\right)\mathcal{D}v - \bar\epsilon\left(t - i\theta\bar\theta\right)\overline{\mathcal{D}}v + 2it\left(\epsilon\bar\theta + \bar\epsilon\theta\right)\dot v\,, \\ \delta\rho &= 4i\alpha\theta\bar\epsilon\rho - \epsilon\left(t + i\theta\bar\theta\right)\mathcal{D}\rho + 2it\left(\epsilon\bar\theta + \bar\epsilon\theta\right)\dot\rho\,, \\ \delta\bar\rho &= 4i\alpha\bar\theta\epsilon\bar\rho - \bar\epsilon\left(t - i\theta\bar\theta\right)\overline{\mathcal{D}}\bar\rho + 2it\left(\epsilon\bar\theta + \bar\epsilon\theta\right)\dot{\bar\rho}\,. \end{aligned} \tag{3.5}$$

The transformations (3.3), (3.4), (3.5) together with the manifest $N=2$ supersymmetry close on $D(2,1;\alpha)$. The $S^1, \bar S_1$ transformations can be restored by commuting (3.5) with (3.4).

Invariant superfield actions consist of a superfield kinetic term and a superpotential. The superfield kinetic term can be easily found to be

$$S_1 = \int dtd^4\theta\left(V^2\right)^{\frac{1}{2\alpha}} = \frac{1+\alpha}{\alpha^2}\int dtd^2\theta\left(v^2 + 4\rho\bar\rho\right)^{\frac{1-2\alpha}{2\alpha}}\left(\mathcal{D}v\overline{\mathcal{D}}v + \mathcal{D}\rho\overline{\mathcal{D}}\bar\rho\right), \tag{3.6}$$

where the $N=4$ superspace integration measure is related to the $N=2$ superspace one by

$$d^4\theta = d^2\theta d^2\xi\,, \qquad \int d^2\theta\,(\theta\bar\theta) = 1\,, \qquad \int d^2\xi\,(\xi\bar\xi) = 1\,. \tag{3.7}$$

The invariance of (3.6) immediately follows from the transformation properties of the integration measure and of V^2, eqs. (2.30), (2.32).

The action (3.6) vanishes for $\alpha=-1$, i.e. for the SU$(1,1|2)$ case, as a consequence of (2.24). Hence, in this situation we consider instead

$$S_{\alpha=-1} = -\frac{1}{2}\int dtd^4\theta\,\left(V^2\right)^{-1/2}\ln V^2 = \int dtd^2\theta\,\left(v^2+4\rho\bar\rho\right)^{-3/2}\left(\mathcal{D}v\overline{\mathcal{D}}v + \mathcal{D}\rho\overline{\mathcal{D}}\bar\rho\right). \tag{3.8}$$

Under $N=4$ superconformal transformations the lagrangian density in (3.8) changes by a total derivative, in contrast to (3.6) where the lagrangian is a tensorial density. The invariance of (3.8) immediately follows from the results of [5] where a similar action was considered for the basic superfield having the same transformation properties as $(V^2)^{-1/2}$ and being subject to constraints more general than (2.24). The field content of [5] was different however.

Let us note that $\frac{1}{1+\alpha}S_1$ is regular for any α and coincides with $S_{\alpha=-1}$ for $\alpha=-1$. In manifestly $N=4$ covariant notation the action (3.8) is recovered from (3.6) in the following way. The latter is invariant for any nonzero α. Let us expand its lagrangian density around $\alpha=-1$:

$$\left(V^2\right)^{\frac{1}{2\alpha}} = \left(V^2\right)^{-\frac{1}{2}-\frac{\alpha+1}{2}+O((\alpha+1)^2)} = \left(V^2\right)^{-1/2} - \frac{\alpha{+}1}{2}\left(V^2\right)^{-1/2}\ln V^2 + O\big((\alpha{+}1)^2\big)\,. \tag{3.9}$$

The first term does not contribute to the action in virtue of (2.24), and so we have

$$\frac{1}{\alpha+1}\left[\left(V^2\right)^{\frac{1}{2\alpha}} - \left(V^2\right)^{-1/2}\right] = -\frac{1}{2}\left(V^2\right)^{-1/2}\ln V^2 + O(\alpha{+}1)\,, \tag{3.10}$$

which yields just (3.8) in the limit $\alpha=-1$.

The potential term is easier to present in the $N=2$ superfield formulation:

$$S_p = \int dtd^2\theta\,\ln\left(\frac{v+\sqrt{v^2+4\rho\bar\rho}}{2}\right). \tag{3.11}$$

To construct this term, one starts from the obvious ansatz

$$S_p^{(0)} = \ln v = u + \text{const.} \tag{3.12}$$

to which S_p should simplify upon reduction to the $N=2$ superconformal mechanics based on the supergroup SU$(1,1|1)$ [5]. The variation of (3.12) with respect to (3.3) reads

$$\delta S_p^{(0)} = -\frac{\delta\,(\rho\bar\rho)}{v^2}\,. \tag{3.13}$$

Thus to have an $N=4$ supersymmetric potential term we have to add a term which compensates for (3.13), namely

$$S_p^{(1)} = \frac{\rho\bar\rho}{v^2}\,. \tag{3.14}$$

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Iterating this recursive procedure, we eventually uncover the final form (3.11) of the $N=4$ invariant potential term via

$$\ln v + \frac{\rho\bar{\rho}}{v^2} - \frac{3}{2}\left(\frac{\rho\bar{\rho}}{v^2}\right)^2 + \frac{10}{3}\left(\frac{\rho\bar{\rho}}{v^2}\right)^3 - \frac{35}{4}\left(\frac{\rho\bar{\rho}}{v^2}\right)^4 + \cdots =$$
$$= \ln v + \sum_{n=0}^{\infty}(-1)^n \frac{(2n+1)!}{(n+1)!(n+1)!}\left(\frac{\rho\bar{\rho}}{v^2}\right)^{n+1} = \ln\left(\frac{v+\sqrt{v^2+4\rho\bar{\rho}}}{2}\right). \quad (3.15)$$

One may check that this expression is invariant, up to a total derivative, with respect to both types of SU(2) transformations (3.4) as well as with respect to the $S^2, \bar{S}_2$ transformations (3.5). Hence, it is $D(2,1;\alpha)$ invariant.

It is interesting to rewrite the potential term in a manifestly $N=4$ supersymmetric form, i.e. in $N=4$ superspace in terms of the superfield V^{ik}. It turns out to be a superconformally invariant version of the standard Fayet-Iliopoulos term $\sim \int dt d^4\theta \mathcal{W}$:

$$S_p = -i\sqrt{2}\int dt d^4\theta \left(\int_0^1 dy\, \partial_y \mathcal{W}\frac{1}{\sqrt{V^2}}\right). \quad (3.16)$$

The scale invariance of S_p is easily seen. Indeed, the integration measure has the dilatation weight (-1) (in units of length), while V^{ik}, V^2 and $\mathcal{W}$ have weights α, 2α and $\alpha+1$, respectively. Then, the weight of the lagrangian density in (3.16) is just $+1$, and (3.16) is scale invariant. It is also easy to show its invariance under the whole $N=4$ superconformal symmetry. One should take into account the transformation law of the superspace integration measure (2.30) and the transformation laws (2.32) and (2.35). The invariance of S_p under the gauge transformations (2.27) of the prepotential $\mathcal{W}$ immediately follows from the constraints (2.24). The extra parameter y in (3.16) defines a deformation $\mathcal{W}(y)$ with $\mathcal{W}(1)=\mathcal{W}$ and $\mathcal{W}(0)=const.$ Hence, the expression (3.16) is analogous to the standard representation of WZW terms on group manifolds via an integral over an extra parameter [28]. In particular, we observe that in the variation δS_p this integral can be taken off (one should use (2.25) while checking this):

$$\delta S_p = -i\sqrt{2}\int dt d^4\theta\, \delta\mathcal{W}\frac{1}{\sqrt{V^2}}. \quad (3.17)$$

Therefore, this term makes a tensor contribution $\sim 1/\sqrt{V^2}$ to the full $N=4$ superfield equations of motion as obtained by varying the total action with respect to the prepotential $\mathcal{W}$. This WZW structure is not accidental: in the bosonic sector the superpotential (3.16) contains the $d=1$ WZW term on the sphere $S^2 = \mathrm{SU}(2)/\mathrm{U}(1)$.

Re-expressing the superpotential (3.16) in the $N=2$ superfield form (3.15) is rather involved though straightforward. One should make use of the relation

$$\mathcal{W} = -\frac{1}{6}\theta_i\bar{\theta}_k V^{ik} + \frac{1}{12}D^i(\theta^2\bar{\theta}^k V_{ik}) - \frac{1}{12}D^i(\theta^2\bar{\theta}^k V_{ik}), \quad (3.18)$$

which is valid modulo gauge transformations from (2.27). Substituting this representation in (3.16) and integrating by parts, one brings (3.16) to the form

$$S_p = \frac{i\sqrt{2}}{24}\int dt d^4\theta \theta^2\bar{\theta}^2 \int_0^1 dy \left[3D_i\bar{D}_k\frac{\partial_y V^{ik}}{\sqrt{V^2}} + \bar{D}^k\frac{D^i\partial_y V_{ik}}{\sqrt{V^2}} - D^k\frac{\bar{D}_i\partial_y V^{ik}}{\sqrt{V^2}}\right]. \quad (3.19)$$

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After this one inserts the relation

$$\theta^2\bar\theta^2 = -4(\theta\bar\theta)\,(\xi\bar\xi)\,,$$

integrates in (3.19) with respect to $\xi, \bar\xi$, expresses all $D_\xi, \bar D_\xi$ derivatives through the $N = 2$ ones $\mathcal{D}, \overline{\mathcal{D}}$ using the relations (2.17), and finally integrates by parts with respect to $\mathcal{D}, \overline{\mathcal{D}}$ in order to remove the delta function $\theta\bar\theta$. Important relations in the course of this calculation are

$$\frac{1}{\sqrt{v^2+4\rho\bar\rho}}(2\rho\mathcal{D}v - v\mathcal{D}\rho) = \mathcal{D}\left[\frac{-2\rho}{\sqrt{v^2+4\rho\bar\rho}(v+\sqrt{v^2+4\rho\bar\rho})}\right] \tag{3.20}$$

and its complex conjugate. The integral with respect to y can be taken off at the end, finally confirming the relation

$$S_p = -i\sqrt{2}\int dt d^4\theta \left(\int_0^1 dy\, \partial_y \mathcal{W}\frac{1}{\sqrt{V^2}}\right) = \int dt d^2\theta \ln\left(v + \sqrt{v^2+4\rho\bar\rho}\right). \tag{3.21}$$

4. The bosonic sector

The bosonic worldline action, with all fermions discarded and the auxiliary field $[\mathcal{D},\overline{\mathcal{D}}]v|$ eliminated by its equation of motion, has the following form,

$$\begin{aligned} S_B &= \mu^{-1}\frac{\alpha^2}{1+\alpha}\,(S_1)_B + \nu\,(S_p)_B \\ &= \int dt \left[\mu^{-1}\alpha^2\, e^u\, \dot u^2 + 4\,\mu^{-1}e^u\frac{\dot\Lambda\dot{\overline\Lambda}}{(1+\Lambda\overline\Lambda)^2} - \frac{1}{4}\mu\,\nu^2 e^{-u} + i\,\nu\frac{\overline\Lambda\dot\Lambda - \Lambda\dot{\overline\Lambda}}{1+\Lambda\overline\Lambda}\right], \end{aligned} \tag{4.1}$$

where μ is a constant of dimension of mass and the parameter ν is dimensionless. It describes the dynamics of a particle with coordinates $u(t), \Lambda(t), \overline\Lambda(t)$ in a three-dimensional target space. In (4.1), the first term is the kinetic term of the dilaton u, the second one is the action of the $\mathrm{SU}(2)/\,\mathrm{U}(1) \sim S^2$ nonlinear sigma model, the third one is the standard dilaton potential, and the fourth one is a WZW term on S^2. The dilaton potential arises after eliminating the auxiliary field from the sum of (3.6) and (3.21), while the WZW term comes solely from (3.21). The strict relation between the dilaton potential and WZW term is required by $N = 4$ supersymmetry.

An equivalent form of the action is achieved by expressing it through standard spherical coordinates r, ϑ, φ via

$$\Lambda = \tan\frac{\vartheta}{2}e^{i\varphi}\,, \qquad e^{u/2} = \frac{1}{\sqrt{2}}\mu r\,, \tag{4.2}$$

$$S_B = \frac{1}{2}\int dt\left[4\alpha^2\mu\,\dot r^2 + \mu r^2\left(\dot\vartheta^2 + \sin^2\vartheta\,\dot\varphi^2\right) - \frac{\nu^2}{\mu\, r^2} + 2\nu\cos\vartheta\,\dot\varphi\right]. \tag{4.3}$$

It is also instructive to rewrite (4.1) in such a way that the potential term has the form typical for 3-dimensional conformal mechanics. One denotes $\mathbf{Y} = (v, \rho, \bar\rho)$ and defines the new cartesian coordinates as $X^i = \mu^{-1}\,Y^i|\mathbf{Y}|^{\frac{1-2\alpha}{2\alpha}}$. With this, (4.1) is rewritten as

$$S_B = \int dt\left[\mu\, g_{ik}(X)\dot X^i\dot X^k - \frac{1}{4\mu}\frac{\nu^2}{|\mathbf{X}|^2} + 2i\nu\frac{\epsilon^{3ik}X^i\dot X^k}{(X^3+|\mathbf{X}|)|\mathbf{X}|}\right], \tag{4.4}$$

where

$$g_{ik}(X) = \delta_{ik} + (4\alpha^2 - 1)\frac{X_i X_k}{|\mathbf{X}|^{\mathbf{2}}} \,. \tag{4.5}$$

From this representation it is clearly seen that only at $\alpha = \pm 1/2$ the kinetic and potential terms of $\mathbf{X}$ coincide with those of the standard 3-dimensional conformal mechanics. At any other nonzero value of α one gets a conformally invariant nonlinear sigma model which is a particular case of the general $1d$ sigma model with $3k$-dimensional target space firstly considered in [21]. The advantage of the form (4.4) is that SU(2) is manifest in all terms except for the WZW one.

To see which kind of geometry the sigma model in (4.1), (4.4) reveals, let us look at the line element corresponding to (4.3) (with $\mu = 1$ for simplicity),

$$ds^2 = 4\alpha^2 dr^2 + r^2 d\Omega^2 \,, \qquad r = \sqrt{2}\, e^{u/2} = |\mathbf{X}| \,, \tag{4.6}$$

where

$$d\Omega^2 = d\vartheta^2 + \sin^2\vartheta\, d\varphi^2 \tag{4.7}$$

is the invariant Killing form on the sphere S^2. This metric belongs to a 3-dimensional cone $C(B)$ over the base manifold $B = S^2$ of radius $\frac{1}{4\alpha^2}$ (see [29] for the relevant definitions). At $\alpha = \pm 1/2$ one recovers flat space $\mathbb{R}^3$ (a cone over S^2 of unit radius), while at any other α one ends up with a curved manifold. It is easy to evaluate the corresponding Ricci tensor and scalar curvature as

$$R_{\vartheta\vartheta} = \left(\frac{1}{4\alpha^2} - 1\right), \qquad R_{\varphi\varphi} = \left(\frac{1}{4\alpha^2} - 1\right)\sin^2\vartheta \,, \qquad R = \left(\frac{1}{4\alpha^2} - 1\right)\frac{2}{r^2} \,, \tag{4.8}$$

and all other components being zero. Thus for $\alpha \neq \pm 1/2$ our manifold is neither Ricci-flat nor of constant scalar curvature. Note that such a conical geometry is typical for the bosonic sectors of superconformal theories in diverse dimensions [29]. The metric in (4.6), in agreement with the general finding of [21], is conformally flat, which can be shown by changing the radial variable as $r = \hat{r}^{\frac{1}{2|\alpha|}}$.

Finally, let us discuss some properties of the S^2 WZW term in (4.1). It is not manifestly SU(2) invariant: under the SU(2)/ U(1) transformations of $\Lambda, \bar{\Lambda}$,

$$\delta\Lambda = b + \bar{b}\,\Lambda^2 \,, \qquad \delta\bar{\Lambda} = \bar{b} + b\,\bar{\Lambda}^2 \,, \tag{4.9}$$

it is shifted by a full time derivative

$$\delta\, \frac{\bar{\Lambda}\dot{\Lambda} - \Lambda\dot{\bar{\Lambda}}}{1 + \Lambda\bar{\Lambda}} = \bar{b}\,\dot{\Lambda} - b\,\dot{\bar{\Lambda}} \,, \tag{4.10}$$

i.e. it is invariant up to an abelian gauge transformation as should be. As was observed for the first time in [30], the WZW term is conformally invariant on its own. Also typical for WZW terms [31, 28], the constant ν needs to be quantized,

$$\nu \in \frac{1}{2}\mathbb{Z} \tag{4.11}$$

(see [22] and refs. therein). Thus the coefficient in front of the superpotential in our $N = 4$ superconformal mechanics should be quantized for topological reasons. As a consequence, the dilaton mass (the coefficient of the dilaton potential in (4.3)) is proportional to the square of an integer.[6] This should be contrasted with the $N = 4$ superconformal mechanics of ref. [5] where no WZW term is present in the bosonic sector and, therefore, no restriction on the strength of the potential term arises.

To further clarify the meaning of the WZW term, let us introduce a constant unit 3-vector C^i, $|\mathbf{C}| = 1$, and rewrite the WZW term in (4.4) (with the factor $2i\nu$ detached) as

$$L_{WZW} = \frac{\epsilon^{ikl}\, C^k\, X^l}{[(\mathbf{X}\cdot\mathbf{C}) + |\mathbf{X}|]\ |\mathbf{X}|}\, \dot{X}^i \equiv \mathbf{A}\cdot\dot{\mathbf{X}}\,. \tag{4.12}$$

Choosing the frame $C^3 = 1$, $C^1 = C^2 = 0$, one reproduces just the WZW term in (4.4). Now one observes that this term is nothing but the coupling of a non-relativistic particle $X^i(t)$ to the potential A^i of a Dirac magnetic monopole, with the singular Dirac string oriented along $\mathbf{X} = -\mathbf{C}$. It is easy to calculate the corresponding magnetic field strength

$$F^{kl} = \partial^k A^l - \partial^l A^k = \epsilon^{klp}\,\frac{X^p}{|\mathbf{X}|^3} \tag{4.13}$$

which is indeed independent of C^i. It is also easy to check that the dependence of the WZW term on the unit vector C^i is reduced to a full time derivative (it is a worldline pull-back of a gauge transformation of A^i) and, hence, the action does not depend on this parameter up to topological effects. This follows from the relation

$$\epsilon^{pij}\, C^i\, \partial^j_C\, (L_{WZW}) = -\partial_t \left[\frac{X^p + C^p|\mathbf{X}|}{(\mathbf{X}\cdot\mathbf{C}) + |\mathbf{X}|}\right]. \tag{4.14}$$

Thus eq. (4.11) is just the Dirac quantization condition.

5. Special bosonic dynamics

In this Section we put $\mu = 1$ for simplicity. In standard spherical coordinates $\mathbf{r} = (x^i) = (r, \vartheta, \varphi)$ our bosonic lagrangian

$$\begin{aligned}\mathcal{L}_B &= \frac{1}{2}\big(4\alpha^2\,\dot{r}^2 + r^2\,\dot{\vartheta}^2 + r^2\sin^2\vartheta\,\dot{\varphi}^2\big) - \frac{\nu^2}{2r^2} + \nu\cos\vartheta\,\dot{\phi} \\ &= \frac{1}{2}\big(4\alpha^2\,\dot{r}^2 + \mathbf{L}^2 r^{-2}\big) - A_0(\mathbf{r}) + A_i(\mathbf{r})\,\dot{x}^i\end{aligned} \tag{5.1}$$

describes an electrically charged particle in a spherically symmetric external electromagnetic field given by

$$E_i = \nu^2\frac{x_i}{r^4} \qquad \text{and} \qquad B_i = -\nu\,\frac{x_i}{r^3}\,. \tag{5.2}$$

[6]Some other arguments why this parameter of the bosonic conformal mechanics should be quantized were adduced in [8].

Note that $A_0{=}\nu^2/2r^2$ is just a central scalar potential while $(A_r{=}0, A_\vartheta{=}0, A_\varphi{=}\nu\cos\vartheta)$ yields a Dirac monopole of magnetic charge $g{=}{-}\nu$. The two potentials are related by $N{=}4$ supersymmetry; this will give rise to special dynamics.

Let us analyze the particle motion in more detail. The angular dynamics is independent of α. Obviously conserved is the canonical momentum

$$p_\varphi = r^2 \sin^2\vartheta\,\dot\varphi + \nu\cos\vartheta =: j \tag{5.3}$$

because φ is a cyclic variable. The ϑ equation of motion

$$\partial_t(r^2\dot\vartheta) = \frac{(j{-}\nu\cos\vartheta)(j\cos\vartheta{-}\nu)}{r^2\sin^3\vartheta} \tag{5.4}$$

still permits the solution $\dot\vartheta=0$ while $\dot\varphi\neq0$ provided that

$$j\,\cos\vartheta = \nu\,, \qquad \text{i.e.} \qquad \cos\vartheta = \text{const}\,, \tag{5.5}$$

which places the motion on a (two-dimensional) cone with opening angle ϑ around the 3-axis. This choice of coordinates simplifies

$$\dot\varphi = \frac{j}{r^2} \qquad \text{and} \qquad l^2 := \mathbf{L}^2 = j^2\sin^2\vartheta = j^2-\nu^2 = \text{const}. \tag{5.6}$$

The radial motion is governed by

$$4\alpha^2\,\ddot r = \frac{j^2}{r^3} \qquad \Longrightarrow \qquad E := \frac12\,4\alpha^2\,\dot r^2 + \frac{j^2}{2r^2} = \text{const}\,, \tag{5.7}$$

which is readily solved by

$$r(\varphi) = \frac{r_0}{\cos\frac{\varphi}{2\alpha}} \tag{5.8}$$

where $r_0 = j/\sqrt{2E}$ is the minimal distance to the origin. The trajectories are unbounded; the polar scattering angle is $\Delta\varphi = 2\alpha\pi$. As usual, the motion is fixed by two data: l (or j or ϑ) and E.

Without the central potential $A_0(r)$ the particles would follow geodesics on the 2D cone inside the 3D cone; the magnetic field merely exerts the constraining force. The repulsive force provided by $A_0(r)$, however, is of a special kind because it gives rise to an enhanced ('dynamical') symmetry, at least in the flat case $\alpha^2{=}1/4$: Rotational invariance implies (à la Noether) the conservation of a canonical angular momentum vector $\mathbf{J}$ which, due to the magnetic potential, differs from the kinematical angular momentum $\mathbf{L}=\mathbf{r}\times\dot{\mathbf{r}}$ via

$$\mathbf{J} = \mathbf{L} + \nu\frac{\mathbf{r}}{r} \qquad \Longrightarrow \qquad \mathbf{J}^2 = l^2+\nu^2 = j^2\,. \tag{5.9}$$

From $\frac{\mathbf{r}}{r}{\cdot}\mathbf{J} = \nu$ we infer again that the motion is constrained to a cone around $\mathbf{J}$, and $\mathbf{L}$ is precessing with angular velocity j/r^2. The equation of motion

$$\ddot{\mathbf{r}} = \nu^2\frac{\mathbf{r}}{r^4} - \nu\frac{\dot{\mathbf{r}}{\times}\mathbf{r}}{r^3} = \nu\frac{\mathbf{J}}{r^3} \tag{5.10}$$

confirms, of course, that $\dot{\mathbf{J}} = \mathbf{r}\times$(equation of motion) $= 0$ irrespective of the central potential. For A_0=0 this system has been analyzed, e.g. in [22] where also hidden symmetries in the absence of an electric potential were discussed. The special form of A_0, however, keeps the acceleration $\ddot{\mathbf{r}}$ in a fixed direction ($\sim \mathbf{J}$) which implies

$$\ddot{\mathbf{r}}\times\mathbf{J} = 0 \qquad \Longrightarrow \qquad \dot{\mathbf{r}}\times\mathbf{J} =: \mathbf{I} = \text{const.} \qquad \Longrightarrow \qquad \dot{\mathbf{r}}\cdot\mathbf{I} = 0 \tag{5.11}$$

meaning that the trajectory is planar ($\perp$ $\mathbf{I}$). This result was already apparent in (5.8) for 2α=1. Hence, the particle follows a special hyperbola. Yet another way to seeing this employs the effective potential

$$V_{\text{eff}} = \frac{l^2}{2r^2} + \frac{\nu^2}{2r^2} = \frac{j^2}{2r^2} \tag{5.12}$$

which, together with $r^2\dot{\varphi} = j$, shows that the projection onto the ϑ=$\frac{\pi}{2}$ plane ($\perp$ $\mathbf{J}$) gives just free motion, i.e. a straight line, with l replaced by j.

To characterize the symmetry enhancement present for our particular central potential A_0 (in combination with the magnetic potential A_i), we compute the Poisson brackets of $\mathbf{J}$=(J_k) and $\mathbf{I}$=(I_k) in flat space (for $\alpha^2 = 1/4$)[7] and find

$$\{J_k, J_l\} = \epsilon_{klm}J_m\,, \qquad \{J_k, I_l\} = \epsilon_{klm}I_m\,, \qquad \{I_k, I_l\} = -2H\epsilon_{klm}J_m\,, \tag{5.13}$$

where $H = \frac{1}{2}(\mathbf{p}^2 + \frac{\nu^2}{r^2})$ denotes the hamiltonian of our system. It is part of the symmetry algebra and conserved with value $H = E$. The relation $\mathbf{I}^2 = 2H\mathbf{L}^2$ may be exploited to directly find the trajectory via

$$\sqrt{2E}\,l\,r\sin\vartheta\cos\varphi = |\mathbf{I}|r\sin\vartheta\cos\varphi = \mathbf{r}\cdot\mathbf{I} = \mathbf{r}\cdot(\dot{\mathbf{r}}\times\mathbf{J}) = \mathbf{L}\cdot\mathbf{J} = \mathbf{L}^2 = l^2 \tag{5.14}$$

which yields $r(\varphi) = \frac{l/\sin\vartheta}{\sqrt{2E}\cos\varphi}$ once more. The algebra (5.13) is very reminiscent of the one generated in the Kepler problem by the angular momentum and the Runge-Lenz vector, in the E>0 case of unbounded motion. Indeed, rescaling I_k reveals an SO(3, 1) symmetry algebra, just like in the latter case.

6. Concluding remarks

In this paper we have presented a new variant of N=4 superconformal mechanics based on a nonlinear realization of the most general d=1, N=4 superconformal symmetry associated with the supergroup $D(2, 1; \alpha)$. In the bosonic sector our version contains on the worldline, besides the dilaton field, two fields parametrizing the two-sphere $S^2 \sim \mathrm{SU}(2)/\mathrm{U}(1)$. This field content suggests that the model can be mapped, by a generalization of the equivalence transformation of [24, 23], onto N=4 $\mathrm{AdS}_2 \times S^2$ superconformal mechanics describing a charged $\mathrm{AdS}_2 \times S^2$ superparticle, and so may be a 'disguised' form of the latter (perhaps, for a special value of α).

The nonlinear realization superfield techniques allowed us to construct the relevant action in a manifestly N=4 supersymmetric way in terms of off-shell N=4 superfields

[7]The magnetic field modifies the canonical brackets: $\{p_k, p_l\} = \epsilon_{klm}B_m$ for the momenta p_k=$\dot{x}_k$.

subject to the constraints proposed in [21]. We also gave an $N=2$ superfield form of the action. There exist two separate invariants which, respectively, extend the kinetic term of the dilaton combined with an $\mathrm{SU}(2)/\mathrm{U}(1)$ nonlinear sigma model as well as the potential term of dilaton. For the latter case, $N=4$ supersymmetry requires the dilaton potential to be accompanied by a $d=1$ WZW term on S^2. The former may be viewed as the potential of a radial electric field, and the latter is interpreted as the coupling to a Dirac magnetic monopole. The coupling constant in front of the WZW term is topologically quantized. It simultaneously defines the dilaton mass, and so the latter is also quantized in the $N=4$ superconformal mechanics under consideration. The target space of the full bosonic sigma model for arbitrary nonzero values of the parameter α turned out to be a cone over S^2. The relevance of such conical geometries to superconformal theories was pointed out in [29]. Furthermore, we have studied the classical dynamics of the bosonic sector of the model. Its three worldline fields are the coordinates of a nonrelativistic massive charged particle moving in the conical $3D$ manifold equipped with a certain electromagnetic background. The particle motion is characterized by an interesting interplay between the electric and magnetic forces which restrict the trajectory to the intersection of a $2D$ cone and a plane. In fact, the special form of the potentials gives rise to a dynamical $\mathrm{SO}(3,1)$ symmetry involving the hamiltonian, just like for unbounded motion in the Kepler problem.

It is worth remarking on the relation to previous studies. In components and in $N=1$ superfields, the $N=4$ superconformal mechanics associated with $D(2,1;\alpha)$ was already discussed in [12] as a special class of $1d$ supersymmetric sigma models [32]. However, no detailed form of $D(2,1;\alpha)$ invariant actions was presented. The $D(2,1;\alpha)$ superconformal mechanics models (actually, the $\alpha=0$ and $\alpha=-1$ special cases) made their appearance also as tools for describing the near-horizon geometry of four-dimensional multi black holes in [13, 14]. Yet, no explicit examples of a superconformal $N=4$ superfield action and/or an $N=4$ superpotential were given there.

Besides the problems of establishing the explicit relationship with the $\mathrm{AdS}_2 \times S^2$ superparticle and of considering the quantization of the presented system (constructing the relevant hamiltonian etc.), there are other directions in which the results of this paper could be extended. First of all, it is interesting to elaborate on alternative nonlinear realizations of $D(2,1;\alpha)$ which could give rise to as yet unknown versions of $N=4$ superconformal mechanics. For instance, we might place all the generators of one $\mathrm{SU}(2)$ in the coset, leaving the second $\mathrm{SU}(2)$ in the stability subgroup, or else break both $\mathrm{SU}(2)$ groups by considering the coset $\mathrm{SU}(2) \times \mathrm{SU}(2)/\mathrm{SU}(2)_{\rm diag}$ or the coset $\mathrm{SU}(2)/\mathrm{U}(1) \otimes \mathrm{SU}(2)/\mathrm{U}(1) \sim S^2 \otimes S^2$. The corresponding versions of $N=4$ superconformal mechanics, if existing, should contain three (in the first two cases) or four (in the third case) physical bosonic fields besides the dilaton in their bosonic sectors. It would be interesting to examine whether there exist suitably constrained $N=4, d=1$ superfields capable of accommodating these field contents. We expect that the precise form of the constraints is predicted by the nonlinear realization formalism, like in the case considered in this paper. In the four-field cases, $d=1$ versions of the hypermultiplet are likely to be recovered in this way.

Another intriguing problem is the construction of $N=8$ superconformal mechanics. It was recently noticed [33] (see also [34, 35]) that one can realize off-shell $N=8, d=1$ Poincaré

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supersymmetry on a $N=4$ chiral superfield and the $N=4$ superfield V^{ik} subject to the constraints (2.19). This representation is a $d=1$ reduction of the vector $N=2, d=4$ multiplet, just like the $N=4$ multiplet described by V^{ik} is a reduction of the $N=1, d=4$ vector multiplet [21]. The corresponding manifold of physical bosons is 5-dimensional. This matches nicely with the bosonic content of the hypothetical $N=8$ conformal mechanics based on a nonlinear realization of one of possible $N=8, d=1$ superconformal groups, viz. $OSp(4^*|4)$ having $\mathrm{SU}(2) \times \mathrm{SO}(5)$ as R-symmetry [19] and containing the $N=4, d=1$ superconformal group $\mathrm{SU}(1,1|2)$ as a supersubgroup [18]. The relevant supercoset should include the dilaton and the four coordinates of the 4-sphere $S^4 \sim \mathrm{SO}(5)/\,\mathrm{SO}(4)$ as the basic Goldstone superfields.

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